Gravitation: Traveling to the outer planets

In this 'virtual' experiment we will send a spacecraft from the Earth to one of the outer planets: Mars, Jupiter, Saturn, or Uranus. We will use the orbital maneuver called the Hohmann transfer to move a spacecraft between two orbits.

Theory:

The basis of this lab is Newton's Law of Universal Gravitation and the laws of Kepler.

Newton's Law of Universal Gravitation

The law of universal gravitation states that the gravitational force between two spherical homogeneous masses is proportional to the product of the two masses and inversely proportional to the separation squared,

$$F_g = G\frac{m_1 m_2}{r^2} \tag{1}$$

where G = 6.6743 x 10^{-11} N m^2/kg^2 is the gravitational constant. The gravitational force is a force of attraction.

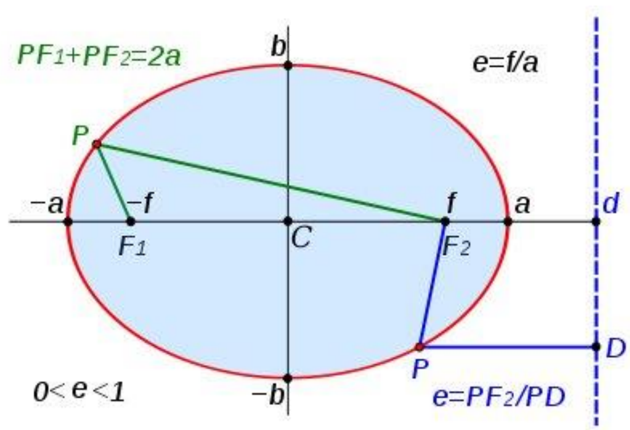
Kepler's Laws

Kepler's Laws are as follows:

1. The orbit of a planet round the Sun describes an ellipse. The Sun is at one of the two foci.

2. The line joining the Sun and the planet sweeps equal areas during equal time intervals.

3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its elliptical orbit.



The semi-major axis is given by ***a*** in the above diagram. The semi-minor axis, focal length, eccentricity are given respectively by ***b***, ***f***, and ***e***. The eccentricity is a number from zero to one and describes how flattened is the ellipse. An eccentricity of 0 gives a circle and an eccentricity of 1 gives a much-flattened ellipse. The eccentricities of the planets in our solar system are all very close to zero; consequently, in this virtual experiment we will suppose that the planets have circular orbits with radius equal to the semi-major axis.

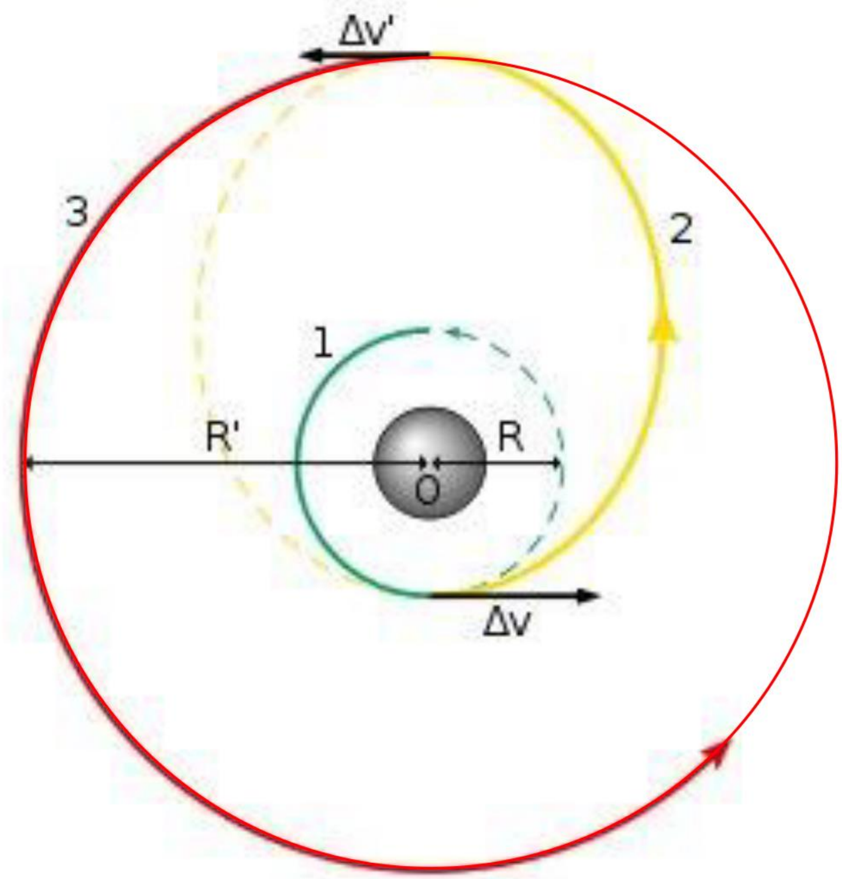
Planetary orbital information can be found at the following website: (not working due to the federal government shutdown)

http://nssdc.gsfc.nasa.gov/planetary/planetfact.html

Use the following link as an alternative:

https://web.archive.org/web/20250818154100/https:/nssdc.gsfc.nasa.gov/planetary/planetfact.html

The following diagram shows a Hohmann transfer orbit.



The Hohmann transfer orbit is characterized by two instantaneous engine impulses that change the momentum (and total energy) of the spacecraft for it to move in an elliptical trajectory (around the sun) from one planetary circular orbit to a different planetary circular orbit. A body in orbit with greater semi-major axis has more energy than in an orbit of smaller semi-major axis. The instantaneous engine impulse leads to a change of velocity called a ***delta-v***. The Hohmann transfer orbit is one of the most fuel-efficient ways (but takes longer) for a spacecraft to move from one orbit to another and is frequently used by NASA.

Calculation of the two *Δvs*

Consider the equation of total energy:

$$E = \frac{1}{2}mv^2 + (-G\frac{Mm}{r}) = -G\frac{Mm}{2a} \tag{2}$$

The potential gravitational energy is given by $U_g = -G\frac{Mm}{r}$ and solving for the velocity v gives

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{3}$$

where the standard gravitational parameter $\mu = GM$. For a circular orbit, when a = r, then this gives

$$v = \sqrt{\frac{\mu}{r}} \tag{4}$$

Consider the nearer circular orbit with r_1 and the farther circular orbit with r_2, then the semi-major axis of the Hohmann transfer orbit will equal $\frac{r_1+r_2}{2}$. Then the total energy and velocity of the spacecraft in the transfer orbit at $r = r_1$ (after the first impulsive burn) using eq. 2 is

$$E = \frac{1}{2}mv^2 + (-G\frac{Mm}{r_1}) = -G\frac{Mm}{2(\frac{r_1+r_2}{2})}$$

$$v = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2r_2}{r_1 + r_2}} \tag{5}$$

Consequently, the first Δv at r_1 is given by the velocity of eq. 5 minus the velocity in the circular orbit at $r = r_1$ of eq. 4, giving

$$\Delta v = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right) \tag{6}$$

Similarly, it can be shown that the $\Delta v'$ for the second impulsive burn is given by

$$\Delta v' = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right) \tag{7}$$

Using Kepler's third law

$$T^2 = \frac{4\pi^2}{\mu}a^3 \tag{8}$$

then the orbital period of the Hohmann transfer orbit T_H with a semi-major axis $\frac{r_1+r_2}{2}$ is given by

$$T_H = 2\pi\sqrt{\frac{(r_1+r_2)^3}{8\mu}} \tag{9}$$

The available launch window dates are related to the synodic orbital period of the planet which you will launch from and the planet that you wish to visit. The synodic period *S* of the Earth–Planet is given by

$$S = \frac{1}{\left|\frac{1}{T_E} - \frac{1}{T_P}\right|} \tag{10}$$

where T_E and T_P are the sidereal periods of the Earth and planet respectively. The synodic period gives the time that you must wait for the next proper orientation of the Earth-Sun-Planet.

Procedure and Interpretation of Results for Outer Planet Travel

1. Using Wolfram Player, draw the circular orbit of the Earth and your assigned outer planet around the Sun in the from and to fields. Assume an eccentricity and orbit inclination of zero. The Sun will have coordinates (0,0). You are to assume that the radius is equal to the semi-major axis of the planet in units of 1×10^6 km.

2. The Hohmann transfer orbit (which is highly elliptical) touches both the Earth orbit and the outer planet orbit at opposite sides of the Sun, which is one of the two focal points of the ellipse.

a. Using the perihelion (the point on the orbit closest to the Sun which corresponds to the Earth-Sun) and aphelion (the point on the orbit farthest to the Sun which corresponds to the outer planet-Sun) of your Hohmann transfer orbit, calculate the coordinates of the second focal point of the transfer orbit. **Hint: distance between the second focal point and aphelion is the Earth-Sun distance**. The Sun is at the first foci at (0,0).

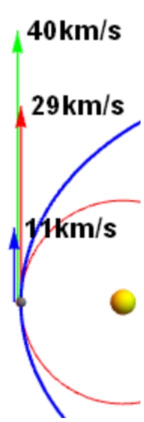
b. In Wolfram Player, enter an x-y coordinate of a point on the Hohmann ellipse and the two foci coordinates. You will enter the first foci (the coordinate of the Sun) and then the calculated second foci, and finally the last point (to define the ellipse) is the perihelion coordinate (radius of the earth orbit) OR the aphelion coordinate (radius of the outer planet). It should look like the Hohmann transfer orbit in the above diagram.

3. Calculate, using **equation 4,** the magnitude of the velocity of the spacecraft in a circular orbit around the Sun with a radius corresponding to the Earth average distance to the Sun. Also calculate the magnitude of the velocity of the spacecraft in a circular orbit around the Sun with a radius corresponding to the orbital radius of your assigned outer planet. **Careful: Use the MKS system, kilogram, meters, and seconds.**

4. Calculate, using **equations 6 and 7,** the two Δ**v** for your Hohmann transfer orbit and the velocity at perihelion and at aphelion of the Hohmann orbit.

Remember: $V_{hohmann}(\text{perihelion}) = V_{earth}(\text{perihelion}) + \Delta V$ and $V_{hohmann}(\text{aphelion}) + \Delta V' = V_{planet}(\text{aphelion})$

5. On your orbital diagram draw (enter the coordinates in Wolfram Player) the initial and final velocities and the delta-v vector at the perihelion and aphelion of the Hohmann transfer orbit. Remember: $V_{hohmann}(\text{perihelion}) = V_{earth}(\text{perihelion}) + \Delta V$ and $V_{hohmann}(\text{aphelion}) + \Delta V' = V_{planet}(\text{aphelion})$



6. Calculate, using **equation 9,** the Hohmann orbit period. Half of the orbit period gives how long it takes for the spacecraft to reach the outer planet. How long does it take for your trip to your planet?

7. a. Using the **trip time** to reach the planet and the **sidereal orbital period** (T_{orb}) of the Earth and planet, calculate the angular displacement of the planet on its orbit for the trip time. Using this angular displacement, determine the initial angular position of the planet at the start of the trip, The Earth-Sun-planet must be aligned in this way so that the spaceship coincides with the planet at the aphelion.

Use:

$\Delta\theta = \omega_{avg}\Delta t$ where $\omega_{avg} = 360°/T_{orb}$ and Δt is the trip time.

$\Delta\theta = \theta_f - \theta_i$

b. Enter the initial angular position (angle) of Earth and your planet in Wolfram Player (clockwise from the left) and show what must be the Earth-Sun-planet alignment on your orbital diagram at the start of the trip.

Note: All planets orbit the Sun in the same direction as the Earth.

8. Calculate, using **equation 10**, the synodic period of your assigned planet – Earth system. Supposing that you are scheduled to launch on Oct. 31, 2025, and you happen to miss this launch window, when will be the next available date to launch?