Projectile Motion

1 Purpose

To show that a given projectile position in its trajectory can be obtained with two different firing angles and to show that for a given angle the trajectory of a projectile is parabolic. For an initial projectile height, the maximum range and the angle which gives the maximum range will be determined.

2 Theory

The trajectory of a projectile where $x_0 = 0$ and $y_0 = 0$ is given by

$$y = (\tan\theta)x - \left(\frac{1}{2}\frac{|a_g|}{{v_0}^2\cos^2(\theta)}\right)x^2 \tag{1}$$

This equation is quadratic in x which gives the shape of a parabola.

The range of a projectile is affected by four parameters: the initial instantaneous speed and initial direction (angle with respect to the horizontal) of the projectile, the initial height, and the acceleration due to gravity. The acceleration due to gravity in the lab is assumed to be $|a_g| = 9.785$ m/s^2.

For an arbitrary initial direction θ at initial projectile height $h = 0$, the horizontal range is

$$R = \left(\frac{{v_0}^2}{a_g}\right)\sin(2\theta) \tag{2}$$

For an arbitrary initial direction θ at initial projectile height h, the horizontal range is

$$R = \left(\frac{{v_0}^2}{a_g}\right)\sin\theta\cos\theta + \sqrt{\left(\frac{{v_0}^2}{a_g}\right)^2\sin^2\theta\cos^2\theta + 2\left(\frac{{v_0}^2}{a_g}\right)h\cos\theta} \tag{3}$$

The firing angle that gives the maximum range for a given initial projectile height h is given by solving $\frac{dR}{d\theta} = 0$. This yields the following:

$$-4\ a_g\ h\sin\theta + 2\ a_g\cos(2\theta)\sqrt{\frac{{v_0}^2\left(8\ a_g\ h\cos\theta + {v_0}^2\sin^2(2\theta)\right)}{{a_g}^2}} + {v_0}^2\sin(4\theta) = 0 \tag{4}$$

3 Procedure

1. Setup the equipment as instructed by your professor.

2. Adjust the time-of-flight pad such that the horizontal distance between it and the launcher is exactly 1.000 m. For an horizontal firing angle of zero degrees (full compression), point the projectile launcher towards the time-of-flight sensor, making sure that the photogate LED at the exit point of the launcher is OFF, fire the projectile. Determine the time of flight of your projectile using the capstone software.

3. Repeat 5 times. Expected time of flights for a distance of 1.000 m is between 0.14 and 0.2 s.

4. Position a small container with opening oriented diagonally towards the launcher approximately 2 to 3 meters from the launcher at a height between 0.5 to 1.5 meters.

5. Determine the low trajectory firing angle to get the projectile into the container.

6. Position a small ring at the top most part of the trajectory such that the projectile moves through the ring on its way to the container. Use a video of the projectile motion through the ring to make that the ring is at the top most part of the trajectory.

7. Use a plumb bob and rulers to measure the geometry of the system as indicated in the following figure. Try to achieve a precision to the millimeter, for example, $y_1 = 88.0\,cm = 0.880\,m$.

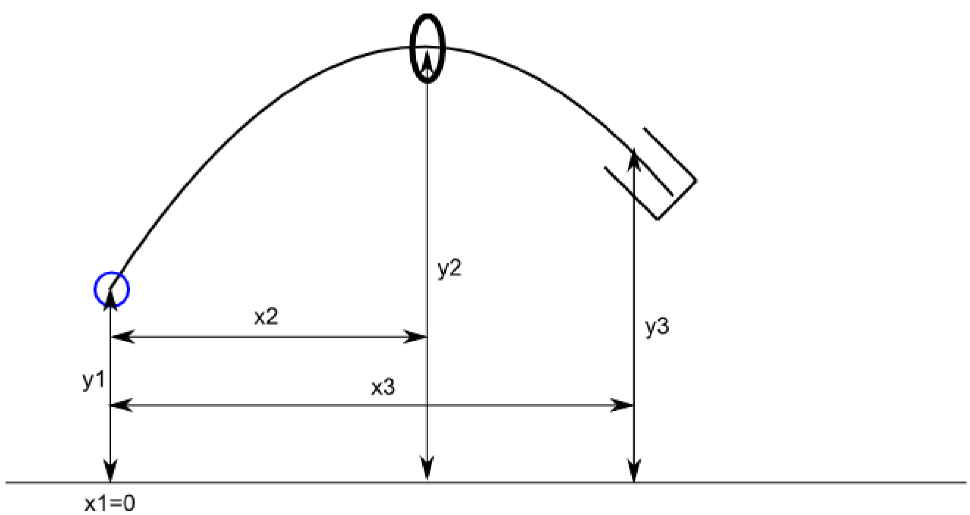

Figure 1: Geometry of the experimental setup

8. Determine the high trajectory firing angle to get the projectile into the container.

9. Place white paper covered with carbon paper on the floor to determine the range of the projectile. Determine the firing angle that gives the maximum range and record that maximum range.

4 Interpretation of Results

1. Determine the initial speed v_0 of the projectile by dividing the horizontal distance $\Delta x = 1.000\ m$ to the time-of-flight sensor by the **average** time of flight, as in $v_0 = \frac{1.000\ m}{t_{avg}}$.

2. Determine the y-component of the initial velocity, v_{0y}, for your low trajectory firing angle using $v_{0y} = v_0 \sin\theta_{low}$, and calculate the maximum vertical displacement Δy of the projectile in its trajectory. Use the third kinematic equation, ${v_y}^2 = {v_{0y}}^2 - 2|a_g|\Delta y$ where v_y is zero at the top of its trajectory.

3. Compare your calculated vertical displacement of interpretation 2 to the experimental vertical displacement: $\Delta y = y_2 - y_1$ according to figure 1. Use the % difference formula to compare.

4. Determine the low and high firing angles grafically.

   - The trajectory equation (Eq. 1) can be manipulated to obtain $2\ y\ ({v_0}^2)\cos^2(\theta) = x\ ({v_0}^2)\sin(2\theta) - |a_g|\ x^2$.
   - For the position of your container ($x = x_3$ and $y = y_3 - y_1$) and your value of v_0 of interpretation 1 and $a_g = 9.785\ m/s^2$, plot two curves on one graph: the left side and right side of the above trajectory equation function of θ. In Excel, generate values from 20 to 80 incrementing 0.1 in the first column. In the second column calculate the left side of the above trajectory equation and in the third column, the right side of the above equation. Plot the 2nd and 3rd column from 20.0° to 80.0° at every 0.1°. The curves should intersect at two points which correspond to your high and low firing angles. Provide a copy of the two rows in Excel where the intersection occurs.
   - Compare using the % difference formula your experimentally and graphically determined firing angles.

5. Using your experimentally determined low and high angles θ_{low} and θ_{high}, v_0, a_g, and y_1, and the equation below, plot the two trajectories of your projectile from $x = 0$ m to 4 meters at every 0.01 m. Use a continuous line plot without data points. Use the trajectory equation (Eq. 1) but **add** the y_1 variable to the right side of the equation: $y = (\tan\theta)x - \left(\frac{1}{2}\frac{|a_g|}{{v_0}^2\cos^2(\theta)}\right)x^2 + y_1$. Indicate the experimental position of the ring and container on the trajectory using a second set of data points and x-y scatter on the same graph.

6. Determine grafically the angle which maximizes the range.

   - The 1st derivative equation = 0 (Eq. 4) can be manipulated to give

$$2\ a_g \cos(2\theta)\sqrt{8{v_0}^2\ a_g\ h\cos\theta + {v_0}^4\sin^2(2\theta)} = 4\ {a_g}^2\ h\sin\theta - a_g\ {v_0}^2\sin(4\theta) \quad (5)$$

- Plot the left side and right side of the above equation, on the same Excel graph, using your values of v_0, a_g, and $h = y_1$ changing the angle θ from 39.0° to 45.0° incrementing 0.1°. The intersection of the two curves gives the angle which maximizes the range. Provide a copy of the two rows in Excel where the intersection occurs. Compare using the % difference formula the experimentally determined and graphically determined values for the angle.
- Using your values of v_0, a_g, $h = y_1$, and the graphically determined maximizing θ, determine the maximum range using equation 3, and compare using the % difference formula your experimentally determined and calculated maximum range.